THE THIRTEENTH REGULAR MEETING OF THE SOUTHWESTERN SECTION.

THE thirteenth regular meeting of the Southwestern Section of the American Mathematical Society was held at the University of Nebraska at Lincoln, on Saturday, November 27, 1920. About twenty persons attended the meeting, including the following members of the Society:

Professor C. H. Ashton, Professor W. C. Brenke, Professor A. L. Candy, Professor C. A. Epperson, Professor M. G. Gaba, Professor E. R. Hedrick, Professor Louis Ingold, Professor T. A. Pierce, Professor H. L. Rietz, Professor W. H. Roever, Professor Oscar Schmiedel, and Professor E. B. Stouffer.

On Friday evening, before the meeting, a smoker was given for the visiting members and their friends, in the University Temple. At this gathering, Professor Hedrick spoke informally concerning the reports of the National Committee on Mathematical Requirements. An interesting discussion followed this informal address. A feature much appreciated by those present was the exhibit of rare mathematical manuscripts and textbooks, by Professor T. J. Fitzpatrick.

At the business session it was decided to hold the next meeting of the Section at the University of Missouri at Columbia. The following committee was appointed:
Professor E. R. Hedrick (Chairman), Professor W. C. Brenke, and Professor E. B. Stouffer (Secretary).

The following papers were presented:

(1) Professor H. J. ETTLINGER: "Existence and oscillation theorems for a system of n differential equations of the second order."

(2) Professor OTTO DUNKEL: "The curve which with its caustic encloses the minimum area."

(3) Professors E. R. HEDRICK, LOUIS INGOLD, and W. D. A. WESTFALL: "Classification of infinities of transformations."

(4) Professor W. C. BRENKE: "On the convergence of certain types of infinite determinants."

(5) Professor LOUIS INGOLD: "Rotation formulas and invariants."

(6) Professor T. A. PIERCE: "Note on Bernoulli's numbers."

(7) Professor OSCAR SCHMIEDEL: "Summation of certain infinite series in finite form. Preliminary report."

(8) Professor M. G. GABA: "A set of postulates for line geometry in terms of line and transformation."

(9) Professor E. B. STOUFFER: "Calculation of the invariants and covariants for ruled surfaces."

(10) Professor E. B. STOUFFER: "Semi-covariants of a general system of linear homogeneous differential equations."

(11) Professor H. L. RIETZ: "Frequency distributions that result from appyling certain transformations to normally distributed variates."

(12) Professor S. LEFSCHETZ: "On certain invariant numbers of algebraic varieties, with application to abelian varieties."

In the absence of the authors, the papers of Professor Ettlinger, Professor Dunkel, and Professor Lefschetz were read by title.

Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. Professor Ettlinger considers the system

$$\frac{d}{dx_i}\left[K_i(x_i, \lambda_1, \lambda_2, \cdots, \lambda_n)\frac{du}{dx_i}\right] - G_i(x_i, \lambda_1, \lambda_2, \cdots, \lambda_n)u = 0$$

$$A_{ij}u_i(a_i) - B_{ij}K_i(a_i, \lambda_1, \lambda_2, \cdots, \lambda_n)\frac{du_i(a_i)}{dx_i}$$

$$\equiv C_{ij}u_i(b_i) - D_{ij}K_i(b_i, \lambda_1, \lambda_2, \cdots, \lambda_n)\frac{du_i(b_i)}{dx_i},$$

where A_{ij}, B_{ij}, C_{ij}, D_{ij} are functions of $\lambda_1, \lambda_2, \cdots, \lambda_n$, and where $i = 1, 2, \cdots n$ and $j = 1, 2$. Under suitable restrictions, the existence of characteristic numbers for the system is proved, and an oscillation theorem established.

2. Necessary conditions for the minimum area between a curve and its caustic were given by Professor P. R. Rider in a paper read at the April meeting of the Chicago Section of the Society. In this paper Professor Dunkel derives conditions for the minimum area which are necessary and sufficient. The properties of the minimizing curve and its caustic are studied, and it is shown that in the general case the first has a single cusp while the second has two, and that the three cusps have parallel tangents. Several cases of end conditions are examined.

3. In this paper Professors Hedrick, Westfall and Ingold continue the investigation of transformations of which an earlier portion was given in a paper at the summer meeting of the Society in Chicago. A classification of the infinities of a transformation is made, and their principal characteristics are discussed, in a manner analogous to the corresponding matter in the theory of functions of a complex variable.

4. It can be shown easily that the infinite determinant $D(a)$ converges to the value zero if the absolute values of its elements a_{ik} form a convergent series. Professor Brenke shows that this remains true if the elements have the form $\lambda + a_{ik}$, where λ is a constant and $\Sigma|a_{ik}|$ converges; also that the infinite secular determinant, whose principal diagonal elements are $x + a_{ii}$, converges for $|x| \leqq 1$, the value being zero for $|x| < 1$.

5. In this paper Professor Ingold extends the application of the formulas of a former paper on "Rotations in a function space" and shows that certain of these formulas give rise to important invariants. The case of the rotations of a system of two vectors is treated in considerable detail.

6. In his note Professor Pierce sets up an explicit formula for the nth Bernoulli number and uses this formula for determining certain factors of the denominators of Bernoulli's numbers.

7. In this paper Professor Schmiedel deals with series principally of the type $1 + 2^a x + 3^a x^2 + \cdots + (m+1)^a x^m$, and seeks to find expressions for the sum valid for all integral values of a and m. Two expressions for each sum are found according as m is positive or negative, when one or the other of the series is convergent. The special case $x = 1$ is not excepted.

Consideration of the sum for a negative being withheld, the paper is given as a preliminary report.

8. Professor Gaba takes for his basis an undefined element called line and an undefined operation on lines called transformation. In terms of line and transformation point and plane are defined. His six postulates, which are theorems of

line geometry, are sufficient to prove as theorems the postulates of Hedrick and Ingold or of Veblen and Young. Independence examples for the postulates are given.

9. In Wilczynski's Projective Differential Geometry of Curves and Ruled Surfaces, calculation is made of a system of invariants and covariants associated with non-developable ruled surfaces. In the present paper, Professor Stouffer obtains the same results by methods involving much less labor. The general system of two linear homogeneous differential equations of the second order is first reduced to a canonical form, the invariants and covariants of this simplified form are then calculated, and the results thus obtained are finally expressed in terms of the coefficients and variables of the original system of differential equations.

10. In a paper read before the Society in April, 1920 (see abstract in this BULLETIN for June, 1920) Professor Stouffer calculated a complete system of seminvariants of a general system of linear homogeneous differential equations of the mth order in n variables. Those results are now extended by the calculation of a complete system of semi-covariants of the same general system of differential equations.

11. In this paper, Professor Rietz gives the results of an investigation into the nature of the frequency distributions that result from applying certain transformations to each variate of a normal distribution. The transformations used are suggested by experience with various systems of variates, some of which are measurements of distances, others of surfaces, and others of volumes. In particular, if the transformation replaces each variate x by a variate $z = kx^n$, where k and n are constants, the frequency function that gives the distribution of z's is found, and some of its interesting properties are established for special values of n.

12. Professor Lefschetz's paper will appear in an early number of the *Transactions*.

LOUIS INGOLD,
Acting Secretary of the Section.

A REMARK ON SKEW PARABOLAS.

BY PROFESSOR GINO LORIA.

(Extract from a letter to Professor D. E. Smith.)

Permettez-moi une remarque relative à deux articles publiés dans le BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY.

Dans le cahier d'Octobre 1918 Mary F. Curtis a établi le remarquable résultat, que toute parabole gauche rectifiable est une hélice; elle s'est servi des formules suivantes pour representer une des courbes dont il s'agit:

$$x = at^3, \qquad y = bt^2, \qquad z = ct; \tag{1}$$

en conséquence l'axe Oz est le tangente à la courbe à l'origine des coordonnées, tandis que le plan $x = 0$ est le plan osculateur; l'axe Oy est donc la normale principale de la courbe et Ox la binormale. Les formules (1) s'appliquent donc à toute parabole gauche, les axes coordonnées étant orthogonaux.

Or tout celà est échappé à M. Hayashi, qui, un an après (BULLETIN, Octobre 1919) a crû nécessaire de traiter la même question de nouveau à l'aide d'une représentation plus compliquée de le (1), ce qui l'a entrainé à des calculs plus longs de ce qu'il est nécessaire. Cette remarque me parait utile, car les calculs du mathématicien Japonnais pourraient faire croire que les formules (1), *les axes étant rectangulaires*, sont applicables *seulement à une classe* de paraboles gauches, tandis qu'on a droit de les *appliquer à toutes*.

Je vous authorise, cher Monsieur, de faire de tout celà, l'usage qui vous semble bon.

GENOA, ITALY,
November 13, 1920.

THE PSEUDO-DERIVATIVE OF A SUMMABLE FUNCTION.

BY PROFESSOR WILLIAM L. HART.

(Read before the American Mathematical Society September 8, 1920.)

Introduction.—In the present paper we shall consider two different types of derivative functions, to be termed *pseudo-derivatives*. These derivatives will be defined for real-valued functions satisfying suitable conditions of summability. The definitions will state certain functional relations which the pseudo-derivatives must satisfy. These relations involve the well-known notion of convergence in the mean, explained in § 1, below. The pseudo-derivative of the first type is considered in § 2, and that of the second type in § 3. The present paper is concerned merely with the definition of these derivatives and with a discussion of a few of their properties. Applications of such derivatives will be considered in a later paper.

All integrals in this paper are taken in the Lebesgue sense. A function f will be termed *integrable* if both f and f^2 are summable in the Lebesgue sense. All functions and variables will be supposed real-valued. The phrase *almost everywhere* will mean *with the exception of at most a set of points of measure zero*. In the discussion below two functions $F_1(x)$ and $F_2(x)$, defined on a set E, will be called the same if they are equal almost everywhere on E. In all equations below, as for example $F_1(x) = F_2(x)$, it will be understood that the equality may cease to hold on some sub-set of E of measure zero.

1. *Convergence in the Mean.*—We shall have occasion to use a slight extension of the notion of convergence in the mean as it has been defined in the case of sequences of functions.* Let $q(s, h)$ be defined for $|h| \leqq h_0$, $h \neq 0$, and for s on a measurable set E, where s may be considered as a single or as an m-partite variable $(s_1, s_2, \cdots, s_m)$. For every h, suppose that $q(s, h)$ is integrable on E.

DEFINITION 1. *The function $q(s, h)$ converges in the mean*

* Cf. Plancherel, *Rendiconti del Circolo Matematico di Palermo*, vol. 30 (1910), p. 292.

to an integrable function $g(s)$ as h approaches zero in case

$$\lim_{h=0} \int_E [q(s, h) - g(s)]^2 ds = 0. \tag{1}$$

We shall say that $g(s)$ is the *limit in the mean* and shall write, for abbreviation,

$$\text{l.m.}_{h=0}\, q(s, h) = g(s). \tag{2}$$

This type of convergence has properties similar to those* possessed by convergence in the mean as defined for sequences. Certain properties of $g(s)$ are listed below; the proofs of the statements will not be given because of their obvious relationship to proofs in the case of sequences of functions.

(*a*) A necessary and sufficient condition that (1) should hold for some function $g(s)$ is that

$$\lim_{h_1,\, h_2=0} \int_E [q(s, h_1) - q(s, h_2)]^2 ds = 0.$$

(*b*) If $g(s) = g'(s)$ and $g(s) = g''(s)$ both satisfy (2), then $g'(s) = g''(s)$.

(*c*) At least one sequence of values $(h_1, h_2, \cdots, h_n, \cdots)$, with $\lim_{n=\infty} h_n = 0$, can be selected so that $\lim_{n=\infty} q(s, h_n) = g(s)$, almost uniformly† for s on E.

(*d*) The function $g(s)$ satisfies the equation

$$\lim_{h=0} \int_E q^2(s, h) ds = \int_E g^2(s) ds. \tag{3}$$

(*e*) Every sequence $(h_1, h_2, \cdots)$ with $\lim_{n=\infty} h_n = 0$, corresponding to which the sequence $[q(s, h_n);\ n = 1, 2, \cdots]$ converges almost everywhere on E, has the property that

$$\lim_{n=\infty} q(s, h_n) = g(s),$$

almost everywhere on E.

2. *The Pseudo-derivative of the First Kind.*—Let $f(x)$ be defined and integrable for values of x on some measurable set E and let $q(x, h) = [f(x + h) - f(x)]/h$. For a given value of h, if $(x + h)$ is not in E, define $f(x + h) = f(x)$. If, for a given h and x, $f(x + h)$ and $f(x)$ are infinities of the same sign, let

* Cf. Plancherel, loc. cit., p. 294.

† Cf. Plancherel, loc. cit., p. 292.

$q(x, h)$ be given any arbitrary value, say zero. Then, for every $h \neq 0$, $q(x, h)$ is integrable on E.

DEFINITION 2. *The pseudo-derivative of $f(x)$ is* $\text{l.m.}_{h=0} q(x, h)$, *provided this limit exists.* It will be denoted by the symbol $f_x(x)$.

As a consequence of Definition 1, $f_x(x)$ is unique if it exists, and satisfies the functional equation

$$\lim_{h=0} \int_E [q(x, h) - f_x(x)]^2 dx = 0. \tag{4}$$

Moreover, at least one sequence $(h_1, h_2, \cdots)$ can be selected as in (c), § 1, such that

$$\lim_{n=\infty} q(x, h_n) = f_x(x), \tag{5}$$

almost uniformly for x on E. Suppose, for example, that E is the set of all irrational numbers on the interval (0, 1) and let $f(x) = x$. Then it is easy to show that $f_x(x) = 1$.

In analogy with the procedure followed in dealing with the classical derivative, let us denote the pseudo-derivative of order n of $f(x)$ by the symbol $f_{x^n}(x)$, and let us define it to be the pseudo-derivative of $f_{x^{n-1}}(x)$, for $n = 2, 3, \cdots$. In the case of a function $f(x; y_1, \cdots, y_k)$, let us define $f_x(x; b_1, \cdots, b_k)$, the partial pseudo-derivative with respect to x at the point $(y_1 = b_1, \cdots, y_k = b_k)$, to be the pseudo-derivative in the sense of Definition 2 of the function $f(x; b_1, \cdots, b_k)$.

In the two theorems which follow, conditions are given under which the pseudo-derivative is equal to the classical derivative.

THEOREM I. *Suppose that the pseudo-derivative $f_x(x)$ exists and that $f(x)$ also possesses a derivative $df(x)/dx$ almost everywhere on E. Then $df(x)/dx = f_x(x)$ almost everywhere on E.*

Since $df(x)/dx$ exists, we have the equation

$$\lim_{h=0} q(x, h) = \frac{df(x)}{dx}, \tag{6}$$

almost everywhere on E. Since (5) and (6) both hold almost everywhere on E, it follows that the theorem is true.

THEOREM II. *Let the set E be the closed interval (a, b) and suppose that a constant $K > 0$ exists such that, for all values of $h \neq 0$ and for all points x on (a, b), $|q(x, h)| \leqq K$. Then,*

if the classical derivative exists at each point of (a, b), *the pseudo-derivative* $f_x(x)$ *exists and* $f_x(x) = df(x)/dx$.

For values $x > b$ and $x < a$, define $f(x)$ by the equations

$$f(x) = f(a) + (x - a)\frac{df(a)}{dx} \quad (x < a),$$

$$f(x) = f(b) + (x - b)\frac{df(b)}{dx} \quad (x > b).$$

Since $f(x)$ is measurable, it is seen that $df(x)/dx$ is also measurable. Moreover, as a consequence of a well-known property* of Lebesgue integrals it is allowable to invert $\lim_{h=0}$ and $\int_{(ab)}$ on the left side of the following equation and to obtain

$$\lim_{h=0} \int_{(ab)} \left[q(x, h) - \frac{df(x)}{dx} \right]^2 dx = 0.$$

Hence, by definition, $f_x(x) = df(x)/dx$.

It is easily verified that Theorem II remains true if, in place of the assumption of the theorem in regard to $df(x)/dx$, we assume that it exists everywhere on (a, b) except at points x on a set H of measure zero.

3. *The Pseudo-derivative of the Second Kind.*—Let us consider a function $f(x, s)$ defined for values of the variable x on a closed interval (a, b) and for values of the m-partite variable $s = (s_1, \cdots, s_m)$ on a measurable set E. In all the discussion below suppose that, for every x on (a, b), $f(x, s)$ is integrable on E and let

$$q(x, h, s) = \frac{f(x + h, s) - f(x, s)}{h}. \tag{7}$$

For a given value of $h \neq 0$, if the point $(x + h)$ is not on (a, b), define $f(x + h, s) = f(x, s)$. For points s at which $f(x + h, s)$ and $f(x, s)$ are infinities of the same sign, let us give $q(x, h, s)$ any arbitrary value, say zero. Then, if $x = x_0$ is on (a, b), it follows that the function $q(x_0, h, s)$, for every $h \neq 0$, is integrable on E.

DEFINITION 3. *The pseudo-derivative of the second kind of* $f(x, s)$ *with respect to* x *at the point* $x = x_0$, *is* $\text{l.m.}_{h=0} q(x_0, h, s)$, *provided this limit exists.* It will be denoted by the symbol $f_x'(x, s)$.

* Cf. de la Vallée Poussin, *Intégrales de Lebesgue*, p. 44.

For brevity, the designation "second kind" will be omitted in the future in this section of the paper in speaking of the pseudo-derivatives $f_x'(x, s)$. As a consequence of Definition 1, if the function $f_x'(x_0, s)$ exists, it is unique and satisfies the functional equation

$$\lim_{h=0} \int_E [q(x_0, h, s) - f_x'(x_0, s)]^2 ds = 0. \tag{8}$$

If $f(x, s)$ is constant with respect to s it is seen that $f_x'(x, s)$ reduces to the classical derivative of a function of the single variable x. Moreover, from (c), § 1, it follows that at least one sequence $(h_n;\ n = 1, 2, \cdots)$ can be selected with $\lim_{n=\infty} h_n = 0$, such that

$$\lim_{n=\infty} q(x_0, h_n, s) = f_x'(x_0, s), \tag{9}$$

almost uniformly for s on E.

In the theorems that follow some useful properties of the pseudo-derivative $f_x'(x, s)$ are considered.

THEOREM III. *Suppose that at $x = x_0$ the pseudo-derivative $f_x'(x_0, s)$ exists, and that at $x = x_0$ the partial derivative (in the classical sense) $\partial f(x_0, s)/\partial x$ exists, finite or infinite, almost everywhere on E. Then it follows that $\partial f(x_0, s)/\partial x = f_x'(x_0, s)$.*

By hypothesis we have, almost everywhere on E,

$$\lim_{h=0} q(x_0, h, s) = \frac{\partial f(x_0, s)}{\partial x}. \tag{10}$$

It is easily verified that the theorem is an immediate consequence of equations (9) and (10).

THEOREM IV. *Let the function $p(s)$ be bounded and measurable on E, and let $F(x, s) = f(x, s)p(s)$. Then, if $f_x'(x_0, s)$ exists, it follows that there exists, likewise, $F_x'(x_0, s) = f_x'(x_0, s)p(s)$.*

The existence of $F_x'(x_0, s)$ follows from the fact that

$$\lim_{h=0} \int_E [q(x_0, h, s)p(s) - f_x'(x_0, s)p(s)]^2 ds = 0,$$

in case equation (8) holds.

THEOREM V. *If $f_x'(x_0, s)$ exists, and if $p(s)$ is any integrable function, the function $H(x) = \int_E f(x, s)p(s)ds$ has a derivative at the point $x = x_0$, given by the expression*

$$\frac{dH(x_0)}{dx} = \int_E f_x'(x_0, s)p(s)ds. \tag{11}$$

Let $\Delta H = [H(x_0 + h) - H(x_0)]$. Then, because of the Schwarz inequality for integrals, it follows that

$$\left|\frac{\Delta H}{h} - \int_E f_x'(x_0, s)p(s)ds\right| = \left|\int_E [f_x'(x_0, s) - q(x_0, h, s)]p(s)ds\right|$$
$$\leqq \sqrt{\int_E p^2(s)ds}\sqrt{\int_E [f_x'(x_0, s) - q(x_0, h, s)]^2 ds},$$

and this expression, because of (8), approaches zero with h. This establishes the equation (11).

Let us represent by $M(x)$ the square root of the quantity $\int_E f^2(x, s)ds$. Then we have the following theorem.

THEOREM VI. *Suppose that, at $x = x_0$, $M(x_0) \neq 0$ and that there exists $f_x'(x_0, s)$. Then, at $x = x_0$, $M(x)$ has the derivative*

$$\frac{dM(x_0)}{dx} = \frac{1}{M(x_0)}\int_E f_x'(x_0, s)f(x_0, s)ds. \tag{12}$$

We shall establish the existence of $dM^2(x_0)/dx$; from this result (12) will follow immediately. We obtain, by use of (7),

$$\frac{M^2(x_0 + h) - M^2(x_0)}{h} = \int_E \frac{f^2(x_0 + h, s) - f^2(x_0, s)}{h}ds \tag{13}$$
$$= h\int_E q^2(x_0, h, s)ds + 2\int_E q(x_0, h, s)f(x_0, s)ds.$$

As h approaches zero, the first term on the right in (13) approaches zero by (*d*), § 1. The second term approaches $2\int_E f_x'(x_0, s)f(x_0, s)ds$ because

$$\left|\int_E [q(x_0, h, s) - f_x'(x_0, s)]f(x_0, s)ds\right|^2$$
$$\leqq M^2(x_0)\int_E [q(x_0, h, s) - f_x'(x_0, s)]^2 ds,$$

which approaches zero with h. Consequently there exists

$$\frac{dM^2(x_0)}{dx} = 2\int_E f_x'(x_0, s)f(x_0, s)ds.$$

In the future suppose that E is a closed interval $c \leqq s \leqq d$. Consider an infinite system of integrable functions

$$I = [p_n(s); \quad n = 1, 2, \cdots],$$

which are unitary and orthogonal on E; that is,

$$(14)\quad \int_E p_i(s)p_j(s)=d_{ij} \quad (d_{ii}=1,\ i=1,\ 2,\ \cdots;\ d_{ij}=0,\ i\neq j).$$

If $f(x, s)$ satisfies the conditions of this section of the paper, let

$$(15)\quad z_i(x) = \int_E p_i(s)f(x, s)ds \qquad (i = 1, 2, \cdots).$$

For brevity we shall term the $z_i(x)$ the *Fourier coefficients* of $f(x, s)$ with respect to the system I and we shall call the vector $\xi(x) = [z_1(x), z_2(x), \cdots]$ the Fourier vector of $f(x, s)$. As an immediate consequence of Theorem V we may state that, if $f_x'(x_0, s)$ exists, the Fourier coefficients have derivatives at $x = x_0$ given by the expressions

$$(16)\quad \frac{dz_i(x_0)}{dx} = \int_E p_i(s)f_x'(x_0, s)ds \qquad (i = 1, 2, \cdots).$$

If $f_x'(x, s)$ exists, let $\eta(x) = [y_1(x), y_2(x), \cdots]$ represent its Fourier vector with respect to the system I. It is seen from (16) that the y_i and z_i are related by the equation

$$y_i(x) = dz_i(x)/dx.$$

Let us suppose in the future that the system I is complete for the class of all integrable functions on E. That is, we assume that there does not exist any integrable function $k(s) \neq 0$ such that $\int_E k(s)p_i(s) = 0$ for $i = 1, 2, \cdots$. Then we shall establish the following theorem.

THEOREM VII. *For a value of x at which $f_x'(x, s)$ exists,*

$$(17)\quad \lim_{h=0} \sum_{i=1}^{\infty} \left[\frac{z_i(x + h) - z_i(x)}{h} - y_i(x)\right]^2 = 0.$$

It has been proved by F. Riesz* that, if $g(s)$ and $h(s)$ are two integrable functions whose Fourier coefficients with respect to the system I are $(a_1, a_2, \cdots)$ and $(b_1, b_2, \cdots)$ respectively, then

$$\int_E g(s)h(s)ds = \sum_{i=1}^{\infty} a_ib_i.$$

From this Riesz formula, it is easily verified that the infinite

* Cf. Plancherel, loc. cit., p. 296.

sum in (17) is equal to

$$\int_E [q(x, h, s) - f_x'(x, s)]^2 ds,$$

so that Theorem VII is an immediate consequence of (8).

Before considering the next theorem, let us note some results which follow directly from the Riesz-Fischer* theorem concerning Fourier constants. If, for every x on (a, b), $f_x'(x, s)$ exists and is zero almost everywhere on E, then, because of (16), the Fourier coefficients of $f(x, s)$ are constants. Consequently, by the Riesz-Fischer theorem, an integrable function $g(s)$ exists such that for every x on (a, b), $f(x, s) = g(s)$. If $\xi(x) = [z_1(x), z_2(x), \cdots]$ is an infinite set of functions defined for x on (a, b) and is such that $\sum_{i=1}^{\infty} z_i^2(x)$ converges for all values of x, then there exists, for every x on (a, b), a unique function $w(x, s)$ with the following properties:

(a) $w(x, s)$ is integrable on E.

(b) $\xi(x)$ is the Fourier vector of $w(x, s)$ with respect to I.

Let $\eta(x) = [y_1(x), y_2(x), \cdots]$ be a second set of functions with the same properties as $\xi(x)$, and let $u(x, s)$ be the function associated with $\eta(x)$ and having the characteristics corresponding to (a) and (b) above. Then, in regard to $\xi(x)$ and $\eta(x)$, we may state the following theorem.

THEOREM VIII. *If, for a certain x' on (a, b), $\xi(x')$ and $\eta(x')$ satisfy* (17), *the function $w(x, s)$ has a pseudo-derivative at $x = x'$ satisfying the equation $w_x'(x', s) = u(x', s)$.*

To establish the theorem consider the expression

$$(18) \qquad \int_E \left[\frac{w(x' + h, s) - w(x', s)}{h} - u(x', s) \right]^2 ds.$$

By the Riesz formula, expression (18) is seen to equal the infinite sum entering in (17) with $x = x'$. Hence (18) approaches zero with h and, by definition, $w_x'(x', s) = u(x', s)$.

Let us now establish an analog of the mean value theorem, for functions possessing pseudo-derivatives.

THEOREM IX. *For every x on (a, b) suppose that $f_x'(x, s)$ exists and that*

$$(19) \qquad \lim_{x'=x} \int_E [f_x'(x', s) - f_x'(x, s)]^2 ds = 0.$$

* Cf. Plancherel, loc. cit., p. 296.

Let us assume, also, that $f_x'(x, s)$ is integrable with respect to (x, s) in the rectangle $(a \leqq x \leqq b, c \leqq s \leqq d)$. Then it follows that, for all points (x, x') on (a, b),

$$(20)\quad f(x', s) - f(x, s) = (x' - x)\int_0^1 f_x'[x + u(x' - x), s]du.$$

In (20), it should be recalled that, for given values of (x, x'), the conventions of this paper permit that the equation should not hold for a set of points s with measure zero.

Let $\xi(x)$ be the Fourier vector of $f(x, s)$ relative to the complete system I. Then it is seen that the derivatives (16) exist and, moreover, that

$$(21)\quad \frac{dz_i(x)}{dx} - \frac{dz_i(x')}{dx} = \int_E [f_x'(x, s) - f_x'(x', s)]p_i(s)ds.$$

On applying the Schwarz inequality for integrals to the right side of (21), it is seen by (19) that the functions $dz_i(x)/dx$ are continuous for every value of x on (a, b). Therefore

$$(22)\quad z_i(x') - z_i(x) = (x' - x)\int_0^1 \frac{dz_i[x + u(x' - x)]}{dx}du$$
$$= (x' - x)\int_0^1 \left[\int_E p_i(s)f_x'[x + u(x' - x), s]ds\right]du,$$

where the last reduction was accomplished with the aid of (16). Let (x', x) be fixed and let us show that the left and the right sides of (20) have the same Fourier coefficients. From this result, as an immediate consequence of the Riesz-Fischer theorem, it will follow that equation (20) is satisfied.

Let $K(x, x', s)$ represent the right of (20). From the theory of Lebesgue integrals* it follows that $K(x, x', s)$ is defined for all values of s on E except possibly for a set E_1 of measure zero. If arbitrary values are given to $K(x, x', s)$ at points in E_1, K becomes an integrable function on the interval E. Let $(a_1, a_2, \cdots)$ be the Fourier coefficients of $K(x, x', s)$ with respect to I. Then we obtain

$$(23)\quad a_i = (x' - x)\int_E p_i(s)\left[\int_0^1 f_x'[x + u(x' - x), s]du\right]ds$$
$$= z_i(x') - z_i(x) \qquad (i = 1, 2, \cdots),$$

* Cf. de la Vallée Poussin, loc. cit., p. 53.

where the last reduction was made by interchanging integrations in (23) and by comparing the result with (22). The interchange was permissible* because of our present hypotheses. Since $[z_i(x') - z_i(x)]$ is the ith Fourier coefficient of the left side of (20), we have completed the proof of the theorem.

In a later paper the author will consider applications of the present results in the theory of functionals whose arguments are summable functions.

UNIVERSITY OF MINNESOTA,
December 15, 1920.

NOTE ON MINIMAL VARIETIES IN HYPERSPACE.

BY PROFESSOR C. L. E. MOORE.

(Read before the American Mathematical Society December 29, 1920.)

1. IT is known that a necessary and sufficient condition that a surface of two dimensions in hyperspace be minimal is the vanishing of the vector mean curvature.† It is the purpose of this note to show that mean curvature of a variety V_m in a space of n dimensions can be defined in the same way and that its vanishing is a necessary and sufficient condition that V_m be minimal. I shall use the absolute calculus, since one of its chief merits is the ease with which invariants can be written down. In fact the very form of an expression shows whether or not it is invariant. Enough vector analysis is used to simplify the form of the expressions.

The variety V_m can be written vectorially in the form

$$y = y(x_1, x_2, \cdots, x_m).$$

Then

$$ds^2 = dy \cdot dy = \sum_{1}^{m} a_{rs} dx_r dx_s.$$

If we write

$$y_r = \frac{\partial y}{\partial x_r},$$

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* Cf. de la Vallée Poussin, loc. cit., p. 53.

†Wilson and Moore, "Differential geometry of two-dimensional surfaces in hyperspace," *Proceedings of the Amer. Acad.*, vol. 52 (1916).

we see at once that

$$a_{rs} = y_r \cdot y_s. \tag{1}$$

Hence we may write

$$dy \cdot dy = \Sigma y_r \cdot y_s dx_r dx_s.$$

Differentiating (1) covariantly with respect to the first fundamental form,

$$\varphi = \Sigma a_{rs} dx_r dx_s, \tag{2}$$

remembering that the first covariant derivative of the coefficients of the fundamental form is zero, we obtain

$$y_{rt} \cdot y_s + y_r \cdot y_{st} = 0,$$

and since this relation holds for any r, s, t, we may write

$$y_{ts} \cdot y_r + y_t \cdot y_{sr} = 0,$$

$$y_{rs} \cdot y_t + y_r \cdot y_{st} = 0.$$

The vector y is a function of $x_1, x_2, \cdots, x_m$. Since the second covariant derivative of a function is symmetric, we have

$$y_{rs} = y_{sr}.$$

From the last four equations we have

$$y_r \cdot y_{st} = 0, \quad r, s, t = 1, 2, \cdots m.$$

These equations show that y_{st} is perpendicular to y_r, and it is therefore perpendicular also to the tangent m-space of V_m.

Since the y_{rs} are vectors lying in the normal space, we can express them as linear functions of mutually perpendicular vectors in that space, which is of $n - m$ dimensions. Thus we find

$$y_{rs} = b_{1|rs} z_1 + b_{2|rs} z_2 + \cdots + b_{(n-m)|rs} z_{n-m}. \tag{3}$$

We have also the following relations:

$$z_i \cdot z_j = 0, \ (i \neq j); \quad z_i \cdot z_i = 1, \quad z_i \cdot y_s = 0. \tag{4}$$

Differentiating these covariantly, we have

$$z_{i|s} \cdot z_j + z_i \cdot z_{j|s} = 0; \quad z_{i|r} \cdot y_s + z_i \cdot y_{rs} = 0. \tag{5}$$

Then, multiplying (3) through by z_i, we have

$$z_i \cdot y_{rs} = b_{i|rs} = - z_{i|r} \cdot y_s.$$

2. *The Second Fundamental Forms.*—The coefficients $b_{i|rs}$ are generalizations of the coefficients of the second fundamental form of a surface in n-space.* They may be taken as the coefficients of $n - m$ fundamental forms which we can combine into a vector fundamental form

$$\psi = \Sigma y_{rs} dx_r dx_s. \tag{6}$$

We now consider the m systems of curves

$$\lambda_i^{(r)} = \frac{dx_r}{ds_i}, \qquad (i = 1, 2, \cdots m)$$

where ds_i is the element of arc along the curve λ_i. Multiplying by the proper factor, we can make the quantities λ_i satisfy the relation

$$\Sigma_r \lambda_i^{(r)} \lambda_{i|r} = 1.$$

We shall now assume that this relation is satisfied. Such systems are called unit systems. The λ's are called the coordinates of the congruence. If the congruences are orthogonal, the coordinates satisfy the relations

$$\Sigma \lambda_{i|r} \lambda_j^{(r)} = \epsilon_{ij},$$

$$\Sigma \lambda_{i|r} \lambda_i^{(s)} = \epsilon_{rs},$$

where $\epsilon_{kl} = 0$, if $k \neq l$, $\epsilon_{kk} = 1$. In terms of the coordinates, we can write the coefficients of the first fundamental form as follows:†

$$a_{rs} = \Sigma_i \lambda_{i|r} \lambda_{i|s}.$$

The coefficients of the second fundamental form can be written as follows:

$$y_{rs} = \Sigma_{i,j} \omega_{ij} \lambda_{i|r} \lambda_{j|s},$$

where ω_{ij} is a vector. The invariant vectors

$$\xi_i = \Sigma_r \lambda_i^{(r)} y_r = \Sigma_r \lambda_{i|r} y^{(r)} \tag{7}$$

are mutually orthogonal unit vectors tangent to the curves whose covariant coordinates are $\lambda_{i|r}$. Solving these equations for y_r and $y^{(r)}$ we find

$$y_r = \Sigma_i \xi_i \lambda_{i|r}, \quad y^{(r)} = \Sigma \xi_i \lambda_i^{(r)}.$$

* See Wilson and Moore, loc. cit., p. 308.

† Wilson and Moore, loc. cit., §13, p. 289, and §§26, 27, pp. 310–311.

We now introduce Ricci's coefficients of rotation*

$$\gamma_{hkl} = \Sigma_{r,s}\lambda_{h/rs}\lambda_k^{(r)}\lambda_l^{(s)}.$$

Solving these equations for $\lambda_{h/rs}$, we find

$$\lambda_{h/rs} = \Sigma_{i,j}\gamma_{hij}\lambda_{i/r}\lambda_{j/s}. \tag{8}$$

Substitution of this value in the covariant derivative of (7) gives

$$\xi_{i/r} = \Sigma_j\omega_{ij}\lambda_{j/r} + \Sigma_{j,l}\gamma_{ijl}\lambda_{l/r}\xi_j. \tag{9}$$

The condition that the congruence λ_i is a congruence of geodesics† is

$$\gamma_{jii} = 0.$$

3. *The Invariants* ω_{ii}. The curvature of a curve of the congruence λ_i which passes through a given point is obtained by differentiating ξ_i with respect to the arc length s_i. Thus we find

$$C_i = \frac{d\xi_i}{ds_i} = \Sigma_r\frac{\partial\xi_i}{\partial x_r}\cdot\frac{\partial x_r}{\partial s_i} = \Sigma\xi_{i/r}\lambda_i^{(r)},$$

which, by (9), reduces to

$$C_i \equiv \omega_{ii} - \Sigma_j\gamma_{jii}\xi_j.$$

The formula shows that the invariant vector ω_{ii} is the normal component of the curvature of the curve of the congruence in question. The tangential component of the curvature is

$$\gamma = \Sigma_j\gamma_{jii}\xi_j.$$

The invariant γ is then the geodesic curvature,‡ since the curvature of the projection on the tangential space is the projection of the curvature.§ If the congruence λ_i is geodesic, $\gamma_{jii} = 0$. Hence the curvature of a geodesic is directed along the normal and is equal to ω_{ii}.

An important invariant connected with any two differential forms is

$$-\Sigma a^{(rs)}y_{rs} = \Sigma a_{rs}y^{(rs)} = \Sigma_{ijkrs}\,\omega_{ij}\lambda_{i/r}\lambda_{i/s}\lambda_k^{(r)}\lambda_k^{(s)} = \Sigma\omega_{kk}.$$

* Ricci and Levi-Civita, "Méthodes de calcul différentiel absolu," etc., *Math. Annalen*, vol. 54, p. 148.

† Ricci and Levi-Civita, loc. cit., p. 148.

‡ Ricci and Levi-Civita, loc. cit., p. 154.

§ Wilson and Moore, loc. cit., p. 320.

Hence the sum of the normal curvature vectors of m mutually orthogonal curves on the spread is constant. We shall define the mean curvature* h by the relation

$$mh = \Sigma\omega_{ii}.$$

4. *Minimal Varieties.*—Applying the preceding formulas, we can now easily derive the condition that the volume of a variety of m dimensions bounded by a variety of $m-1$ dimensions, chosen arbitrarily, shall be a minimum. The vector element of volume is

$$P dx_1 dx_2 \cdots dx_m = \frac{\partial y}{\partial x_1} \times \frac{\partial y}{\partial x_2} \times \cdots \times \frac{\partial y}{\partial x_m} dx_1 dx_2 \cdots dx_m.$$

The condition for a minimum is

$$\delta \int (P \cdot P)^{\frac{1}{2}} dx_1 dx_2 \cdots dx_m = \int \frac{\delta P \cdot P}{(P \cdot P)^{\frac{1}{2}}} dV = \int \delta P \cdot M dV = 0,$$

where M is the unit tangent n-space

$$M = \frac{P}{(P \cdot P)^{\frac{1}{2}}}.$$

Now

$$\delta P = \frac{\partial \delta y}{\partial x_1} \times \frac{\partial y}{\partial x_2} \times \cdots \times \frac{\partial y}{\partial x_m} + \frac{\partial y}{\partial x_1} \times \frac{\partial \delta y}{\partial x_2} \times \cdots \times \frac{\partial y}{\partial x_m} + \cdots + \frac{\partial y}{\partial x_1} \times \frac{\partial y}{\partial x_2} \times \cdots \times \frac{\partial \delta y}{\partial x_m}.$$

The condition for a minimum then becomes the vanishing of n vector integrals. One of these integrals, after an integration by parts and the omission of the part which vanishes at both limits, becomes

$$\int \left(\frac{\partial \delta y}{\partial x_1} \times \frac{\partial y}{\partial x_2} \times \cdots \times \frac{\partial y}{\partial x_m} \right) \cdot M dV = -\int \delta y \times \frac{\partial}{\partial x_1} \left(\frac{\partial y}{\partial x_2} \times \frac{\partial y}{\partial x_3} \times \cdots \times \frac{\partial y}{\partial x_m} \cdot M \right) dV.$$

*This is the direct generalization of mean curvature as used in the theory of surfaces. There is, however, a mean curvature definable in terms of the first fundamental form, and this is the one used by Einstein. The definition used in this note depends on the space in which V_m lies.

The complete condition then becomes

$$\int \delta y \times \left[\frac{\partial}{\partial x_1}\left(\frac{\partial y}{\partial x_2} \times \cdots \times \frac{\partial y}{\partial x_m} \cdot M\right) - \frac{\partial}{\partial x_2}\left(\frac{\partial y}{\partial x_1} \times \frac{\partial y}{\partial x_3} \times \cdots \times \frac{\partial y}{\partial x_m} \cdot M\right) + \cdots \pm \frac{\partial}{\partial x_m}\left(\frac{\partial y}{\partial x_1} \times \cdots \times \frac{\partial y}{\partial x_{m-1}} \cdot M\right)\right] = 0.$$

Since δy is arbitrary,

$$\frac{\partial}{\partial x_1}\left(\frac{\partial y}{\partial x_2} \times \frac{\partial y}{\partial x_3} \times \cdots \times \frac{\partial y}{\partial x_m} \cdot M\right) - \frac{\partial}{\partial x_2}\left(\frac{\partial y}{\partial x_1} \times \frac{\partial y}{\partial x_3} \times \cdots \times \frac{\partial y}{\partial x_m} \cdot M\right) + \cdots \pm \frac{\partial}{\partial x_m}\left(\frac{\partial y}{\partial x_1} \times \cdots \times \frac{\partial y}{\partial x_{m-1}} \cdot M\right) = 0,$$

which on multiplying out reduces to

$$\frac{\partial y}{\partial x_2} \times \frac{\partial y}{\partial x_3} \times \cdots \times \frac{\partial y}{\partial x_m} \cdot \frac{\partial M}{\partial x_1} - \frac{\partial y}{\partial x_1} \times \frac{\partial y}{\partial x_3} \times \cdots \times \frac{\partial y}{\partial x_m} \cdot \frac{\partial M}{\partial x_2} + \cdots = 0.$$

Now substituting

$$\frac{\partial y}{\partial x_r} = y_r = \Sigma_k \xi_k \lambda_{k|r},$$

$$M = \xi_1 \times \xi_2 \times \cdots \times \xi_m,$$

$$\frac{\partial M}{\partial x_i} = \frac{\partial \xi_1}{\partial x_i} \times \xi_2 \times \cdots \times \xi_m + \xi_1 \times \frac{\partial \xi_2}{\partial x_i} \times \xi_3 \times \cdots \times \xi_m + \cdots,$$

and using (9), the condition for a minimum becomes

$$(\Sigma_i \omega_{ii}) |\lambda_{i|r}| = 0.$$

The determinant $|\lambda_{i|r}|$ is equal to $\sqrt{a}$, where a is the determinant of the first fundamental form, for from the expression $a_{rs} = \Sigma_i \lambda_{i|r} \lambda_{i|s}$ we have

$$a = |a_{rs}| = |\Sigma_i \lambda_{i|r} \lambda_{i|s}| = |\lambda_{i|r}| \cdot |\lambda_{i|s}| = |\lambda_{i|r}|^2.$$

Hence the condition for a minimum is $\Sigma \omega_{ii} = 0$.

The vanishing of the mean curvature vector is a necessary and sufficient condition that the variety be minimal. This is the same condition as that for a 2-surface in hyperspace.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY,
November 24, 1920.

NOTES ON ELECTRICAL THEORY.

BY PROFESSOR H. BATEMAN.

1. *The Production of Light.*—According to an idea that is generally associated with the names of Faraday and J. J. Thomson, energy is radiated from an electric charge in the form of light or electric waves when the lines of electric force issuing from the charge assume a wavy form. This idea has been made more definite by a study of the form of the lines of force in various types of electromagnetic fields. It is known now that waves on the lines of force may be produced either by an oscillation of the electric charge or by the continual emission of either electric or magnetic doublets. The emission of a single electric charge produces a kink in each line of force while the emission of a single magnetic charge produces a rotation of each line of force. When charges of opposite signs are successively emitted in the same direction so as to be equivalent to a doublet, the successive kinks are in opposite directions and are thus equivalent to a solitary wave. When a succession of magnetic poles of one sign are emitted in one direction, the lines of force rotate about this direction but they rotate with different angular velocities. The same is true when magnetic poles of opposite signs are fired out in opposite directions.

If a magnetic pole emitted in one direction is followed by a magnetic pole of the opposite sign so as to give a magnetic doublet, the lines of force rotate first in one direction and then in the other returning finally to their original directions. The result again is that a solitary wave runs along each line of force, but just as in the electrical case one line of force is unaffected.

The question now arises whether two types of light production really occur in nature. This is a question that has been bothering physicists for some time and opinion is divided. An argument in favor of the dual nature of light may be presented as follows:

2. *The Reaction of an Electric Charge upon a Field which Alters its Motion.*—It is usually assumed in electromagnetic theory that when an electric charge A is accelerated by an

external electromagnetic field E, the field E is modified indirectly only by the action of the electron's field on the sources of the field E. This reaction is, however, an after effect and it seems more in accordance with the principles of mechanics to suppose that the acceleration of A is accompanied by a direct local modification of the field E. Of course the acceleration of A produces a modification of A's field and it may be thought that this is the only local modification that occurs, but this modification spreads out in all directions while A's motion is changed only in one direction. At any rate the hypothesis that the field E is locally modified is worth considering; it may be that the disregard of this modification has been the cause of the present difficulties in the electromagnetic theory of radiation.

To get an idea of the nature of this local modification we shall adopt a definite hypothesis with regard to the structure of a line of electric force as viewed by an imaginary observer who is able to distinguish particles that are extremely small and exceedingly close together.

We shall suppose that a line of electric force issuing from a stationary positive charge P is built up of a series of doublets arranged as follows:

$$P+\ -+\ -+\ -+\ -+\ A\ -+\ -+\ -+\ -+\ \rightarrow$$

These doublets may be supposed to be moving in the direction of the arrow with the velocity of light. The free charge at P is only temporarily at rest, for it may be supposed to secure a negatively charged partner from a doublet which arrives at P and move away with the velocity of light, leaving the positive constituent of the doublet at P until this charge in its turn secures a negative partner. The charge at P may be really moving all the time along a zig-zag path in the neighborhood of the point P, the mean free path being comparable in size with the so-called "diameter" of the positive charge. If the mass of the charge is proportional to the number of collisions with doublets per unit time it is easy to understand why the mass is inversely proportional to the "diameter" or mean free path of the free electric charge, especially if the free charge always moves with the velocity of light in describing its zig-zag path. If n denotes the number of collisions per second and m the mass of the positive charge, a lower limit to the value of n may, perhaps, be given by the equation $mc^2 = hn$,

where c denotes the velocity of light and h, Planck's constant. The value of n for the nucleus of a hydrogen atom is then of order 10^{23}. So also if an electron is similarly constructed the number n is of order* 10^{20}. Let us now suppose that an electric charge of the same magnitude as one of the constituents of a doublet meets the line of electric force at A. If the charge is positive it may be supposed to seize the negative constituent of the doublet on the right leaving the positive constituent free. The effect is the same as if the free positive charge had been moved to the right, *i.e.* as if it had been repelled by the free positive charge at P. As for the newly formed doublet, it may be supposed to fly away in some direction which depends, perhaps, on the direction of motion of the free charge which entered at A. It is possible that the colliding charges are broken up into fragments which are scattered in all directions. This is a possibility that ought to be considered but at present it is simpler to assume that the charges remain intact.

In the latter case the result of the encounter between the free electric charge A and the line of force is that the free charge is displaced along the line of force, one of the doublets belonging to the line of force is annihilated and another doublet is formed and emitted in some direction which is probably different from that of the line of force. Mathematically the annihilation of one doublet and the emission of another may be represented by the emission of two doublets one of which travels in the direction of the doublet that was annihilated and is of such a nature as to completely annul the effect of this doublet. Where the former doublet would have a positive charge the emitted doublet must have an equal negative charge.

Electromagnetic fields in which pairs of doublets are emitted from a moving point have already been studied.† They may be derived from the elementary type of field in which charges which are equal but opposite in sign are emitted in different directions.

When the electric charge which meets the line of force at A

* If a free charge moves with the velocity of light, the mean length of the free path in the case of an electron may be of order 10^{-10}. This is about the order of magnitude of the radius of the ring electron. A free charge moving along a zig-zag path may be supposed to behave something like a ring electron.

† *Proc. London Math. Soc.*, (2), vol. 18 (1919), p. 95.

is negative, it may be supposed to seize the positive constituent of the doublet on the left, leaving the negative constituent behind. The result is equivalent to a displacement of the free charge towards the free positive charge at P just as if the negative charge were attracted by the positive charge. There is also a breaking up of a doublet belonging to the line of force and the emission of a new doublet in some direction. The displacement or acceleration of the free electric charge in the direction of the line of force may be regarded as an elementary process of which the equations of motion of an electric charge are a consequence. The free charge of an electron may be supposed to encounter at least 10^{10} lines of force from different charges in 10^{-10} seconds and the resultant acceleration may consequently be proportional to a mean value of the electric force in this interval of time. It is difficult to derive formally the equations of motion from our hypothesis regarding the structure of a line of force; the difficulty may arise from the fact that the true line of force of the familiar type of electromagnetic field is a limit of the discontinuous line of force which we have pictured in order to make the argument clear. The discontinuous structure may, however, be more nearly correct than the continuity with which we are so familiar. The idea of mass being proportional to a frequency of collision seems reasonable when compared with other physical laws such as the law of mass action in chemistry and Planck's relation between an energy quantum and frequency. The idea arose originally in connection with a theory of gravitation which has already been sketched elsewhere.*

3. *The Lines of Force of an Oscillating Electric Pole.*—It is known that a line of electric force of a moving electric pole can be considered as the locus of a series of light particles projected from the pole at successive instants.† If $\boldsymbol{v}$ denotes the velocity of the pole at time τ and the unit vector $\boldsymbol{s}$ indicates the direction of projection of the light particle emitted at this instant, $\boldsymbol{s}$ satisfies the differential equation

$$(c^2 - v^2)\boldsymbol{s}' = c\boldsymbol{s} \times \boldsymbol{v}' \times (c\boldsymbol{s} - \boldsymbol{v}),$$

where primes denote differentiations with respect to τ. When

* *Messenger of Mathematics*, vol. 48, Aug., 1918, p. 74.

† Leigh Page, *Amer. Journ. of Sci.*, vol. 38, Aug., 1914, p. 169; H. Bateman, this BULLETIN, March, 1915.

the motion is rectilinear and along the axis of z the above equation is easily solved. If (l, m, n) are the three components of $\boldsymbol{s}$, we have*

$$l = \frac{2\beta\sqrt{c^2 - v^2}\cos\alpha}{c + v + \beta^2(c - v)}, \qquad m = \frac{2\beta\sqrt{c^2 - v^2}\sin\alpha}{c + v + \beta^2(c - v)},$$

$$n = \frac{c + v - \beta^2(c - v)}{c + v + \beta^2(c - v)},$$

where α and β are constants for each line of force.

It is easy to see that n increases with v and is stationary when v is stationary. Let us consider the case of a simple periodic oscillation about the origin in which $v = ap\cos p\tau$. The extreme values of n are then given by $v = \pm ap$ and occur when the electric pole is at the origin. Let P be the position at time t of a light particle shot out in one extreme direction; then remembering that l', m', and n' are zero when the electric pole is at the origin, we see that as a point moves from P along the figures of the line of force at time t, the increments of the coordinates are

$$dx = -cld\tau, \qquad dy = -cmd\tau, \qquad dz = (v - cn)d\tau,$$

where $v = \pm ap$.

The direction of the tangent at P is found to be the same whether we take the upper or the lower sign in the expression for v. Let us draw a line OQ parallel to the tangent and study the relation of the line of force to this radial line OQ. The line of force evidently has a wavy form and P is the crest of one of the waves. The distance of P from the radius OQ is easily found to be

$$\frac{2\beta uc(t - \tau)}{[c^2(1 + \beta^2)^2 - u^2(1 - \beta^2)^2]^{1/2}},$$

where $u = \pm ap$. As we move from crest to crest going away from O, the height of a crest above OQ increases very slowly if u is small. Thus if we are considering the flight of a light particle, the height of a crest increases approximately at a rate

$$\frac{2\beta u}{1 + \beta^2},$$

* F. D. Murnaghan, *Amer. Journ. of Math.*, April, 1917; Johns Hopkins Circular, July, 1915.

which is equal to u when $\beta = \pm 1$ and is less than u for other values of β. When $\beta = 0$ the line of force is straight.

If the intensity of the ordinary light emitted from an oscillating electric pole depends on the square of the acceleration ap^2, then for a given intensity $u = ap$ is inversely proportional to p, so that for light of high frequency u may be very small indeed and the crests on a wavy line of force may be very close to the radial line OQ even at a great distance from the pole. It may be on this account that light of high frequency behaves much more nearly as if it were corpuscular in nature than light of low frequency.

The intensity of the light depends upon the slope of the line of force away from OQ rather than upon the distance of the crests from OQ. When v is very small in comparison with c the light vector at a great distance from O may be taken to be equal to es'/cr, where e is the electric charge associated with the pole and r is the distance from the pole.

4. *Lines of Electric Force in Some Other Types of Electromagnetic Fields.*—If we suppose that an emission of light resulting from an acceleration of an electron is accompanied by an emission of doublets owing to a local modification of the field which accelerates the electron, there is some uncertainty as to the way in which we should represent mathematically the resulting electromagnetic field when the light is of the frequency of visible light or even X-rays, for a large number of doublets may be emitted in a single period. It seems best at first to simplify matters by considering first fields in which doublets or charges are emitted in all directions and then some fields in which doublets and charges are emitted only in a few particular directions.

Let us first of all consider the field $(\boldsymbol{E}, \boldsymbol{H})$ given by

$$\boldsymbol{H} + i\boldsymbol{E} = \nabla\sigma \times \nabla\tau + \frac{i}{c}\left[\frac{\partial\tau}{\partial t}\nabla\sigma - \frac{\partial\sigma}{\partial t}\nabla\tau\right],$$

where $r = c(t - \tau)$ is the distance of a point P whose coordinates are (x, y, z) from a moving point Q whose coordinates at time τ are (ξ, η, ζ) and whose velocity is $\boldsymbol{v}$. The quantity σ is given by the equation

$$\sigma = \frac{(\boldsymbol{l}\cdot\boldsymbol{r}) + k}{(\boldsymbol{v}\cdot\boldsymbol{r}) - cr},$$

where $\boldsymbol{l}$ is a complex vector which is a function of τ and k is a function of τ. The vector QP is denoted by $\boldsymbol{r}$.

In this type of field both electric and magnetic charges are emitted in all directions from the moving pole Q and the electric charge associated with the pole may vary. When $\boldsymbol{l} = e\boldsymbol{v}'$ and $k = e(c^2 - v^2)$, we obtain the field of a moving electric pole from which no charges are emitted. If $k = e(c^2 - v^2)$ and the vector function $\boldsymbol{l}$ changes sign periodically or represents a rotating vector the electric and magnetic charges which are emitted may be regarded as forming doublets.

The lines of electric force may be considered as made up of light-particles projected from the pole Q when k is real. The differential equation satisfied by the unit vector $\boldsymbol{s}$ which indicates the direction of projection at time τ is

$$k\boldsymbol{s}' = \boldsymbol{p} + (\boldsymbol{s} \times \boldsymbol{q}) - \boldsymbol{s}(\boldsymbol{s} \cdot \boldsymbol{p}),$$

where

$$\boldsymbol{p} + i\boldsymbol{q} = c\boldsymbol{l} + i(\boldsymbol{v} \times \boldsymbol{l})$$

and both $\boldsymbol{p}$ and $\boldsymbol{q}$ are real vectors.

We are interested in the case when a line of force returns periodically to its original form. Let us consider uniform circular motion and write $\boldsymbol{v} \equiv (v \cos \omega\tau,\ v \sin \omega\tau,\ 0)$, $\boldsymbol{l} \equiv (-e\omega v \sin \omega\tau, e\omega v \cos \omega\tau, ib)$; then

$$\boldsymbol{p} \equiv [-(b + ec\omega)v \sin \omega\tau, (b + ec\omega)v \cos \omega\tau, 0],$$

$$\boldsymbol{q} \equiv (0, 0, cb + e\omega v^2).$$

It is easy to see that if $\boldsymbol{p} \neq 0$ the differential equation for $\boldsymbol{s}$ possesses only a finite number of solutions of type

$$\boldsymbol{s} = \frac{\alpha + \beta \cos \omega\tau + \gamma \sin \omega\tau}{f + g \cos \omega\tau + h \sin \omega\tau}$$

and that there are consequently only a finite number of lines of force which return to their original form after a period of time $2\pi/\omega$. In the case when $\boldsymbol{p} = 0$ and $k = e(c^2 - v^2)$ the differential equation for $\boldsymbol{s}$ reduces simply to

$$\frac{d\boldsymbol{s}}{d\tau} = \omega(\boldsymbol{n} \times \boldsymbol{s}),$$

where $\boldsymbol{n}$ is a unit vector in the direction of the axis of z. All

the solutions of this equation are periodic with a period $2\pi/\omega$ and so all the lines of force return to their original form periodically. The field in this case is of the type considered by Leigh Page.*

As an example of a field in which charges are fired out in particular directions let us consider the following expressions for the components of $\boldsymbol{E}$ and $\boldsymbol{H}$:

$$E_x = \frac{ex}{r^3} + \frac{\lambda y}{x^2+y^2} - \frac{\nu zx}{r(x^2+y^2)},$$

$$E_y = \frac{ey}{r^3} - \frac{\lambda x}{x^2+y^2} - \frac{\nu zy}{r(x^2+y^2)}, \qquad E_z = \frac{ez}{r^3} + \frac{\nu}{r},$$

$$H_x = \frac{\lambda zx}{r(x^2+y^2)} + \frac{\nu y}{x^2+y^2}, \qquad H_y = \frac{\lambda zy}{r(x^2+y^2)} \cdot \frac{\nu x}{x^2+y^2},$$

$$H_z = -\frac{\lambda}{r},$$

where λ and ν are functions of $\tau = t - r/c$.

We may write $\boldsymbol{E} = r d\boldsymbol{s} - c\boldsymbol{s}d\tau$, where $\boldsymbol{s}$ is a unit vector with components (l, m, n) which at time τ is in the direction of the radius vector r, provided

$$\frac{dl}{d\tau} = \frac{c\nu}{e}\frac{ln}{l^2+m^2} - \frac{c\lambda}{e}\frac{m}{l^2+m^2},$$

$$\frac{dm}{d\tau} = \frac{c\nu}{e}\frac{mn}{l^2+m^2} + \frac{c\lambda}{e}\frac{l}{l^2+m^2},$$

$$\frac{dn}{d\tau} = -\frac{c\nu}{e}.$$

These equations give

$$\frac{d}{d\tau}(l+im) = \frac{c\nu}{e}\frac{n(l+im)}{1-n^2} + \frac{ic\lambda}{e}\frac{l+im}{1-n^2},$$

consequently when n has been expressed in terms of τ, $l + im$ may be found by a simple integration. It may be noticed that when $\nu = 0$ and λ is a constant the lines of force revolve around the axis of z at different rates. If θ denote the angle which a line of force makes with the axis of z and ϕ denote

* *Proc. Nat. Acad. of Sci.*, March, 1920, p. 115.

its angular velocity around this axis we have $\sin^2\theta\cdot\dot{\phi} = c/e$. If each line of force carried a unit mass at unit distance from the origin the angular momentum of the mass would thus be the same for all the lines of force.

An attempt made previously* to obtain the lines of force when any system of charges or doublets is emitted from a moving pole is vitiated by an unfortunate oversight. It appears that Z cannot be made equal to unity as was assumed. The differential equations to be solved are consequently of type

$$g(\sigma, \bar{\sigma})\frac{d\sigma}{d\tau} = f'(\bar{\sigma}, \tau), \qquad g(\sigma, \bar{\sigma})\frac{d\bar{\sigma}}{d\tau} = f'(\sigma, \tau),$$

where g is a function† whose form is independent of the emitted system and consequently independent of the form of f'.

CALIFORNIA INSTITUTE OF TECHNOLOGY,
October, 1920.

SHORTER NOTICES.

Table de Charactéristiques de Base 30030, *donnant en un seul coup d'oeil les facteurs premiers des nombres premiers avec* 30030 *et inférieurs à* 901,800,900. By ERNEST LEBON. Tome I, Premier Fascicule. Paris, Gauthier-Villars, 1920.

To one who has spent eight years of his life in making a factor-table for the first ten millions the plan to extend such a table to the limit 901,800,900 seems like a rather serious undertaking. If such a table were constructed according to the plan devised by Burckhardt and employed by Dase and Glaisher the number of pages would exceed one hundred thousand, and with five hundred pages to the volume would fill some two hundred volumes! If, as in the tables published by the Carnegie Institute, the multiples of 2, 3, 5 and 7 were omitted, and the pages somewhat enlarged to take care of the large divisors certain to appear, the number of pages would be

* *Proc. London Math. Soc.*, (2), vol. 18 (1919), p. 123, *Phil. Mag.*, vol. 34, Nov., 1917, p. 419. In the notation used in the last paper the equations should be

$$g(\alpha, \beta)\frac{d\alpha}{d\tau} = f(\beta, \tau), \qquad g(\alpha, \beta)\frac{d\beta}{d\tau} = f(\alpha, \tau).$$

† In the case of a stationary pole, $g = (1 + \sigma\bar{\sigma})^{-2}$.

42,943, and the work would appear in 86 immense volumes! M. Lebon's table omits multiples of 2, 3, 5, 7, 11, 13; but a table of smallest divisors omitting such multiples, and constructed according to the extremely condensed plan of the Carnegie tables would still run into some 34,594 pages, or 69 large volumes!

Furthermore, M. Lebon notes the inconvenience of tables which give only the smallest divisors, and in his table proposes to list all the divisors! Such a plan, followed out in factor-tables of the kind already published, would certainly multiply their bulk by two, and probably by five. This Table de Charactéristiques is to take the place of three or four hundred volumes of the most compactly arranged factor-tables yet invented! If the author were not listed on the title page as Agrégé de l'Université, Professeur honoraire de Mathématiques au Lycée Charlemagne, Lauréat de l'Académie Francais et de l'Académie des Sciences; if he were not able to cite favorable mention of it in the important mathematical societies of France, Italy, Spain and America; if the persons presenting his work before the various learned societies were not men like Volterra, Appell, Rouche, Neuberg, Nielsen, Darboux; if he had not been granted medals and prizes in recognition of his work, one would be inclined to dismiss the undertaking as absurd. With such a list of illustrious men and associations behind it one must take the publication seriously.

Of the 56 pages of tables contained in the Premier Fascicule of Tome I the first 40 are devoted to numbers of the form $Bk + 1$, where $B = 30030$, and give the factors of all such numbers for values of k not greater than B. The table is not arranged with increasing values of k, however, but with increasing values of I where the number $Bk + 1 = I \cdot I'$. The factorization of both I and of I' is given, and the corresponding value of k is also listed. This table of 40 pages not only serves to give the factors of all numbers not larger than B, but is also used, as we shall see, in finding the factors of larger numbers.

The second table, called a "Table of Characteristics," gives the factorization of numbers of the form $Bk + 1$ for k less than B, arranged with increasing values of k. The Premier Fascicule carries this table up only as far as $k = 4680$. There are 14 pages in this table and when complete up to $k = B$ it will cover some 90 pages altogether, unless the increasing number and size of the factors necessitate wider spacing be-

tween the columns. This table, then, when complete, will serve to give the complete factorization of all numbers of the form $Bk + 1$ within the limits proposed for the table. A similar table, giving the factors of all numbers of the form $Bk + I$, where I is any one of the 5760 numbers less than, and prime to B, would complete the factor table, but as such an extension of the table would mean 518,400 pages, or over a thousand large volumes, M. Lebon has another plan to propose. He undertakes to throw any number of the form $B\kappa + I$ into the form $Bk + 1$. This is indeed not difficult. Multiply the number $n = B\kappa + I$ by I' where $I \cdot I' = Bk + 1$. An easy reduction gives $nI' = BK + 1$ where $K = \kappa I' + k$. If, now, K is less than B we can find the factorization of nI' in the table of 90 pages noted above, I' being obtainable for any I in the first little table of 40 pages. But since I' may have any value less than B, and κ also may have any value less than B it is clear that K may have any value less than $B \times B = 901{,}800{,}900$, "Avec le Tome II commencera la table des charactéristiques $K > 30029$." It is worth while to pause to examine the extent of this "Tome II."

It is not quite true that K takes all values less than this limiting number, because those which give primes may be omitted. The number of pages necessary for the whole table will be, however, approximately B times the number necessary for the table as far as $K = B$, that is to say 90 pages. A simple multiplication gives then, 2,702,700 as the approximate number of pages in "Tome II." Truly a stupendous volume! Equal to 5,405 volumes of 500 pages each!

The reviewer believes, from his own experience in such matters that one page a day for the computation alone would be exceedingly rapid work. Allowing 300 working days for the year, "Tome II" will be completed some 9,000 years from now! Those who are looking forward eagerly to the appearance of this volume will be pained to note that M. Lebon has received from the Academy a notice "qui provoque un arrêt dans les calculs":

"Le Conseil d'Administration, d'accord avec la Commission technique, a décidé, dans sa dernière séance, qu'il y avait lieu d'attendre, pour continuer ses subventions, que la partie déjà exécutée de votre travail fût imprimée."

D. N. LEHMER.

Newton. By GINO LORIA. Rome, A. F. Formíggini, 1920. 69 pp.

THIS little volume, written by one who seems never to rest in his literary labors, is number 52 of a series of biographies issued under the rather poetic title of Profili. The title is quite appropriate, for each number is a kind of side glance at the profile of the individual whose biography is rapidly sketched by some worthy and skilled literary artist.

What Professor Loria has done is to set forth in popular style the incidents in Newton's life that are more or less known to mathematicians but are not so familiar to the general cultured public. It is needless to say that he has not attempted to present material not already known, since this was not his problem. In this case his is the mission of a popularizer. He has apparently drawn, directly or indirectly, from Brewster's well-known work, as all other historians of mathematics have done for the last two generations.

The life of Newton is briefly told,—his rather unpromising boyhood, the unusual promise shown by him at Trinity College, Cambridge, his power of easily grasping the theories of his predecessors, his rise to fame, his discovery of the laws of gravitation and of the fluxional calculus, and his later contributions to mathematics in general. The writer calls attention to the fact that the respect due to Newton is partly a case of hero worship, not less marked than the mental attitude of devout pilgrims to the Holy Sepulchre when in the presence of a portion of the true cross. In this he gives us the view of one who sees the Anglo-Saxon civilization from without, and it is a fair question whether we who see it from within have not been guilty of unduly exalting the contributions and powers of the author of the Principia. The fact is that Newton is today a good deal of a mystery, and there is need for a mathematical scholar of judicial mind, of literary ability, and of sufficient leisure to give us a new biography of the discoverer of the fluxional calculus, and a bibliography that is worthy of the man. We have Brewster and De Morgan and various minor writers, but we need an authoritative life of Newton and a definitive edition of his works, with a list, approximately complete, of published Newtoniana in general. It is very strange that we have modern editions of the works of the great mathematicians of Germany, France, and Italy, and of British scholars like Cayley and Sylvester,

but that Newton seems so sacrosanct that we feel that we must consult his works only in a first or second edition of two centuries ago, or in the Opera quae exstant omnia which Horseley published in 1779–1785. As an example of our lack of information, we do not really know why it is that Newton left Cambridge and spent the last thirty years of his life in London. The usual reasons are familiar, but no one of them seems sufficient to account for this substantially complete break with Cambridge; and if one will seek to ascertain, for example, what Newton was doing in 1720, he will very likely find an absolute blank. Did the story of his serious mental failure have any foundation in fact? We do not know from any authoritative evidence. We can say that on such a year he contributed an article on a certain subject, that at such a time he was corresponding with one of the Bernoullis, that he was president of the Royal Society during such a period, and we can link these facts together in a fragmentary fashion; but some capable historian is needed to take the problem up, basing his solution upon trustworthy documentary evidence and giving to the world a biography that shall be worthy of such a genius.

Needless to say that Professor Loria has not, in sixty-nine pages, attempted anything of this kind; but when he speaks of the state of hero worship that exists in all our Anglo-Saxon minds, with respect to Newton, he does us a good service. It is not a question of Newton's greatness,—it is a question of the precise facts by the knowledge of which we can properly measure that greatness.

It must not be thought that Professor Loria is an iconoclast, —far from it. Voltaire, in one of his baser moods, wrote that he had once believed that Newton was made the master of the mint because of his great merit, but that he "avait une nièce assez aimable, nommée Madame Conduit; elle plut beaucoup au grand trésorier Halifax. Le calcul infinitésimal et le gravitation ne lui auraient servi de rien sans une jolie nièce." If Professor Loria had a biased mind, he might have repeated the story without comment, but he shows, as we all know, that the statement was false and was unworthy the better nature of the "old invalid of Ferney."

The famous priority controversy relating to Newton and Leibniz is considered briefly, but the facts are now fairly well known and the dispute no longer has its former interest.

The Profili now sell for three lira each in Italy,—about fifteen American cents, at the present rate of exchange. They are artistically printed and each is the work of a scholar. It seems strange that we have never, in this country, been able to support a series of this kind. The chief criticism of the work of this Italian press lies in the number of typographical errors that appear. On a single page (68), for example, we have the "Leiters of Sis Isaac Newton," "R. Benthey" (for Bentley), "and other autenthic documents," "Eeral of Macclesfield, "Côtes" (for Cotes), and "Comptes-Rendus," while on page 66 there are no less than nine errors of a similar nature.

DAVID EUGENE SMITH.

Mathematiker Anekdoten. Zweite, stark veränderte Auflage. Von W. AHRENS. Teubner, Leipzig and Berlin, 1920. 42 pp.

WHETHER the telling of an anecdote shall provoke the interest of a pleased smile or the different amusement which leads to a shrug of the shoulders depends intimately and delicately upon the mental associations which arise involuntarily when a story is related; and the latter in turn depend upon the varied elements, and even the most minute, which make up the daily life and experience and environment. Hence it has always been, and perhaps always will be, difficult for one people to appreciate the humor of another. It is therefore natural that a book of anecdotes, containing humorous ones among others, shall be addressed by an author principally to his own countrymen.

These stories related by Ahrens of mathematicians and things mathematical are evidently intended primarily for his own countrymen; hence it is fitting that far the greater space should be given to men and things that are German. One of the pleasing features of the booklet is the inclusion of fifteen or more excellent likenesses of mathematicians. The stories range in excellence from some of high quality to some which are not pleasing. We do not find much of value in the story of the boys who convinced a simple old man that in their use of logarithm tables they were mastering the house numbers of Europe. We are only mildly interested when we are told of L. Fuchs' surprise when a long computation in his lecture led to the result $0=0$, that he first painfully suspected

an error, but that he then regained his composure and said "'Null = Null' ist ja ein sehr schönes und richtiges Resultat." But the story of the youth of Gauss will please every one who enjoys the activity of genius. An effective impression of the progress of mathematical instruction is made by a brief account of mathematical instruction in the "good old times." The account of the self-taught Arago's successes when examined on one occasion by Louis Monge and on another by Legendre is inspiring; any one who does not know these stories will be repaid if he looks up the booklet for them alone.

R. D. CARMICHAEL.

Leçons sur les Fonctions automorphes. Par GEORGES GIRAUD. Gauthier-Villars, Paris, 1920. 126 pp. [Collection de Monographies de Emile Borel.]

IN a course of lectures at the College of France the author of this monograph has treated several aspects of the theory of automorphic functions. His central object was to present from a single point of view properties of several sorts of automorphic functions as investigated by many authors following the lead of Poincaré and Picard. The present volume contains an exposition of a part of these lectures.

The plan of treatment proposed places certain limitations upon the choice of material and its method of organization. No attempt is made to give a complete exposition of the theory of the fuchsian functions of Poincaré. Of the topics not treated the following may be mentioned: the theorems on the representation of the coordinates of algebraic curves; those on the integration of linear differential equations with algebraic coefficients and regular singular points; generalizations of these for functions of more than one variable; the theory of the so-called Kleinian functions of Poincaré. A treatment of certain of the omitted topics, it is said (page 4), will be given later in the form of a memoir or perhaps in the form of another volume similar to the present one.

The author's principal purpose of unification is realized through the use of a certain general class of groups (Γ) satisfying certain hypotheses (H) of a rather general character. In the first instance the groups are given, not explicitly but only implicitly through their possession of certain properties. The postulational basis of this treatment is laid at the beginning

of the first chapter (pages 8–11); it is too long for reproduction in the review. The whole treatment proceeds in intimate dependence upon this logical basis. Certain central results are first obtained in association with any group for which hypotheses *H* are satisfied.

Chapter II is devoted to a class of linear groups: for the case of one variable these became the fuchsian groups whose properties have been treated by Poincaré; for the case of two variables they become the hyperfuchsian groups of Picard; for the case of more than two variables they are groups which have been studied by Fubini. All the groups of the class are shown to satisfy hypotheses *H* of the first chapter. Further properties of this special class are developed. In Chapters III and IV there is a similar treatment of certain quadratic groups, and of certain groups formed from a set of several given groups. The treatment in these four chapters (pages 8–90) is general and abstract in character and is intimately dependent upon the basic postulates *H*. The final Chapter V (pages 91–123) is devoted primarily to the functions of Poincaré. On account of the special features of the more restricted theory, certain results become more precise than in the more general theory; and this fact is brought out by a derivation of the detailed results.

R. D. CARMICHAEL.

Archimedes. By SIR THOMAS LITTLE HEATH. Society for the Promotion of Christian Knowledge, London, 1920. vi + 60 pp.

AFTER becoming familiar with the larger works which have made Sir Thomas Heath so widely known, the reader who takes up the little work under review will do so with a feeling of surprise. The academic world has come to expect from his pen only such extended treatises as he has written upon Apollonius of Perga, Diophantus, Aristarchus, Euclid, and Archimedes,—treatises filled with erudition and written in that classical style of which he is a master. If the reader is a man of the cloister, the surprise will be unpleasant; if he is a man among men, it will be the opposite. Since the spirit of the time makes scholars more and more men of the world, the balance of judgment is certain to be in favor—shall we say of the appellant or the defendant?

What Sir Thomas Heath has done is to give a brief and

popular summary of the results which he has set forth in his well known treatise on the great Syracusan, and in his pamphlet on the recently discovered manuscript (1906) on the method of treating mechanical problems. He has done this by means of seven brief and easily read chapters as follows: I. Archimedes; II. Greek Geometry to Archimedes (substantially the subject of the first volume of his forthcoming history of Greek mathematics); III. The Works of Archimedes; IV. Geometry in Archimedes; V. The Sandreckoner; VI. Mechanics; and VII. Hydrostatics. To this summary he has added a brief bibliography and a chronological table.

The story is told in a style that will easily appeal to the popular taste,—that is, to a taste that has not been trained to enjoy the more severe intellectual food served at the tables of erudition. The opening chapter, on the life of Archimedes, will fascinate any youthful reader and will be read with pleasure by those of more mature years. The chapter on Greek geometry requires a little knowledge of algebra and geometry, but can easily be read by any one with a high school education. The third chapter sets forth briefly the nature of the extant works of Archimedes and gives a list of those which are now known only by title. The chapter on geometry in Archimedes reaches a little further into the domain of mathematics, but will give the college freshman, in brief space, the essential information which he needs with respect to the contributions of the Greeks to the calculus. The sandreckoner is within the easy reach of the high school student, but the chapters on mechanics and hydrostatics will be found, in view of the denatured courses in physics in our American high schools, beyond the average pupil.

The frontispiece is from David Gregory's edition of Euclid (Oxford, 1703), already used in the author's work on Diophantus, but for some reason (perhaps the size of the page) there is no indication of the interesting source.

As a piece of technical bookmaking the work is, of course, not in the same class with the other publications of the author. The products of but few publishing houses can compare with those of the Cambridge and Oxford presses. Moreover, the book is published for popular use, and expense is a weighty consideration. The Society for the Promotion of Christian Knowledge has done and is doing a praise-worthy work in placing such works within the reach of every purse. It is a matter of great regret that we have not, in this country, a simi-

lar foundation which enables us to publish a series of works of this nature at a nominal price, as is done in several European countries. No better field than this is today open in this country for establishing a relatively small foundation which should seek to satisfy a hunger for good reading. With the work of this British society in mind, one can readily excuse the lack of an index, and the poor paper which war conditions have imposed.

DAVID EUGENE SMITH.

The Theory of the Imaginary in Geometry together with the Trigonometry of the Imaginary. By J. L. S. HATTON, principal and professor of mathematics, East London College. Cambridge, England, University Press, 1920. 216 pp. and 96 figures.

THE word theory in the above title is to be understood in a very non-technical sense. Indeed, apart from the idea of the invariant elements of an elliptic involution on a straight line, no theory is found at all. The purpose of the book is rather to furnish a certain graphical representation of imaginaries under a number of conventions more or less well known. Three concepts run through the work: first, an incompletely defined idea of the nature of an imaginary; second, the analogy with the geometry of reals; third, the use of coordinate methods, assuming the algebra of imaginaries.

Given a real point O and a real constant k, an imaginary point P is defined by the equation $OP^2 = -k^2$. The two imaginary points P and P' are the double points of an involution having O for center, and ik for parameter. The algebra of imaginaries is now assumed, and a geometry of imaginary distances on a straight line is built upon it. The reader is repeatedly reminded that in themselves there is no difference between real and imaginary points; that differences exist solely in their relations to other points. In the extension to two dimensions both x and ix are plotted on a horizontal line, while y and iy are plotted on a vertical line. Imaginary lines are dotted, and points having one or both coordinates imaginary are enclosed by parentheses, but otherwise the same figures are used for proofs, either by the methods of elementary geometry, or by coordinate methods.

In the algebra of segments it is shown that an imaginary distance $O'D'$ can be expressed in the form iOD, wherein OD is a real segment, or at most by OD times some number. Now

follows a long development of the extension of cross ratios, etc., to imaginaries. In fact every word of this is found implicitly in any treatment of the invariance of cross ratios under linear fractional transformation.

In Chapter II the conic with a real branch is introduced, beginning with involutions of conjugate points on lines having imaginary points on the conic. If the coefficients in the equation of a circle are real, the usual graph of $x^2 + y^2 = a^2$ for real x and real y is followed by replacing y by iy, then proceeding as before. The former locus is called the (1, 1) branch, and the latter the $(1, i)$ branch of the circle. Similarly, it has a $(i, 1)$ branch, and another, (i, i), but the latter has no graph. This idea is applied in all detail to ellipses, hyperbolas, and parabolas; in the case of the central conics it is also followed by replacing rectangular coordinates by a pair of conjugate diameters. The ordinary theorems of poles and polars, and the theorems of Pascal, Brianchon, Desargues, Carnot are shown to apply. Indeed, after having established the applicability of cross ratios in the earlier chapters, all these proofs can be applied in the same manner as to reals, without changing a word.

Imaginary angles are brought in in Chapter III. The reasoning by analogy has no meaning when applied to minimal lines. It is not shown why the results should be definite and unique in all other cases, but the statements on pages 71 and 73 are restricted by an undefined "in general." On page 73 it is tacitly assumed that the angle between the two bisectors of an imaginary angle is a right angle, and it is stated as a theorem later on that the sum of all the imaginary angles on one side of a straight line, measured at a point on the line, is two right angles. An imaginary line may rotate about a real line from a position of coincidence to one of perpendicularity with the real line. "A right angle may be divided into a series of imaginary angles." On page 70 we are told that "the point at infinity on the line is of the same nature as the base point. It can be regarded as real or imaginary." On page 91 we find: "the sine of θi increases from 0 to $i\infty$." After some skirmishing, trigonometry of imaginary angles is treated by the usual exponential formulas.

A curious mixture of premises introduces the critical lines, that is, those passing through the circle points. It is by no means clear that these are the only exceptions to be made in the application of the preceding theory.

The most satisfactory part of the entire treatise is that founded on the analytic basis. Here the reason for the exceptional nature of the critical lines becomes at once evident. The book could have been materially improved if the direct contradiction met in attempting to measure either distances or angles concerned with these lines had been pointed out.

Notwithstanding the preceding remarks, the book under review is not without merit. It is plainly and consistently written, the development frequently involves considerable skill, the results include a large number that are distinctly worth while, and the exercises furnish mental gymnastic material for many a lesson. The reviewer must contend, however, that it does not furnish a theory of imaginaries.

VIRGIL SNYDER.

COMMENT ON A PREVIOUS REVIEW.

In Professor Wilson's review of Eddington's *Space, Time and Gravitation*, which appeared in a recent number of this BULLETIN, he expresses his doubt as to the accuracy of Eddington's statement that Riemann never dreampt of a physical application of his analysis. That this doubt is amply justified and that Eddington's statement shows a lack of acquaintance with certain portions of the historical background of the general relativity theory, may be readily shown by a brief quotation from Riemann's famous Habilitationsschrift *Uber die Hypothesen welche der Geometrie zu Grunde liegen.* (We make use of Clifford's translation.)

"Either, therefore, the reality which underlies space must form a discrete manifoldness, or we must seek the ground of its metric relations outside it, in binding forces which act upon it. The answers to these questions can only be got by starting from the conception of phenomena which has hitherto been justified by experience, and which Newton assumed as a foundation, and by making in this conception the successive changes required by facts which it cannot explain. . . . This leads us into the domain of another science, of physics, into which the object of this work does not allow us to go to-day."

This statement (made in 1854) foreshadows the work of Einstein, and thus furnishes further evidence, if further evidence be deemed necessary, of the remarkable insight into mathematical and physical questions possessed by Riemann.

CHARLES N. MOORE.

NOTES.

At the Chicago meeting of the American Association for the Advancement of Science, Professor E. H. Moore was elected president for the next two years. Dr. R. S. Woodward, ex-president of the Carnegie Foundation and of the American Mathematical Society, was reelected treasurer, and Dr. B. E. Livingston permanent secretary, each for a term of four years. Professor Oswald Veblen was elected chairman of Section A (mathematics), and a vice-president of the Association. The winter meeting of 1921 will be held at the University of Toronto.

At the sixth annual meeting of the Mathematical Association of America, held at Chicago, December 28–29, the following officers were elected: president, Professor G. A. Miller; vice-presidents, Professors R. C. Archibald and R. D. Carmichael; members of the Board of Trustees, to serve until January, 1924, Professors A. A. Bennett, Florian Cajori, H. L. Rietz, and D. E. Smith. Professor C. F. Gummer was chosen to fill the vacancy on the Board caused by the election of Professor Carmichael as vice-president.

On the occasion of the mathematical congress held in Strasbourg in the summer of 1920, the following mathematicians were elected honorary members of the Society of Sciences, Agriculture, and Arts of the Lower Rhine: Professor L. Crelier, of the University of Bern; Professor L. E. Dickson, of the University of Chicago; Professor G. Koenigs, of the Sorbonne; Professor J. Larmor, of Cambridge University; Professor N. E. Nörlund, of the University of Lund; Professor E. Picard, of the Sorbonne; Professor C. de la Vallée Poussin, of the University of Louvain; Professor V. Volterra, of the University of Rome.

Announcement has just been made that Professor Dunham Jackson will present the principal paper of a symposium on the general theory of approximation by polynomials and trigonometric sums on Friday afternoon, March 25th, at the meeting of the American Mathematical Society at the University of Chicago. The meeting of the Society will extend over Saturday, March 26th.

On account of the meeting of the Central Association of Teachers of Mathematics and Science in St. Louis during Thanksgiving week of this year, the officers of the Southwestern Section of the American Mathematical Society and the officers of the Missouri Section of the Mathematical Association of America have decided to hold the meetings of these organizations also in St. Louis, at Washington University, instead of at the University of Missouri, as had been announced previously.

The opening (January) number of volume 22 of the TRANSACTIONS OF THE AMERICAN MATHEMATICAL SOCIETY contains the following papers: *Arithmetical paraphrases*, by E. T. Bell; *The construction of algebraic correspondences between two algebraic curves*, by Virgil Snyder and F. R. Sharpe; *Concerning certain equicontinuous systems of curves*, by R. L. Moore; *Fundamental systems of formal modular seminvariants of the binary cubic*, by W. L. G. Williams; *A property of two* $(n+1)$*-gons inscribed in a norm-curve in n-space*, by H. S. White; *Recurrent geodesics on a surface of negative curvature*, by H. M. Morse; *On the location of the roots of the jacobian of two binary forms, and of the derivative of a rational function*, by J. L. Walsh; *On functions of closest approximation*, by Dunham Jackson.

The concluding (October) number of volume 42 of the AMERICAN JOURNAL OF MATHEMATICS contains the following papers: *Geometrical significance of isothermal conjugacy of a net of curves*, by E. J. Wilczynski; *Observations weighted according to order*, by P. J. Daniell; *Some determinant expansions*, by L. H. Rice; *A general implicit function theorem with an application to problems of relative minima*, by K. W. Lamson; *On the Laplace-Poisson mixed equation*, by R. F. Borden; *Characteristic subgroups of an abelian prime power group*, by G. A. Miller.

The December number (vol. 22, no. 2) of the ANNALS OF MATHEMATICS contains the following papers: *The mean of a functional of arbitrary elements*, by Norbert Wiener; *On certain determinants associated with transformations employed in thermodynamics*, by J. E. Trevor; *The permanent gravitational field in the Einstein theory*, by L. P. Eisenhart; *On the structure of finite continuous groups with a finite number of exceptional infinitesimal transformations*, by S. D. Zeldin; *Conformal*

mapping of a family of real conics upon another, by T. H. Gronwall; *On the location of the roots of the derivative of a polynomial*, by J. L. Walsh.

The January number (vol. 14, no. 1) of THE MATHEMATICS TEACHER announces the change in the management of that journal, which will be hereafter the official organ of the newly-formed National Council of Teachers of Mathematics. The new staff of editors is constituted as follows: J. R. Clark, editor-in-chief; E. R. Smith, assistant editor; and Alfred Davis, H. D. Gaylord, Marie Gugle, and J. W. Young. This staff will be assisted by an advisory board of twenty members under the chairmanship of Dean W. H. Metzler, formerly editor-in-chief of the same journal.

The March number of the AMERICAN MATHEMATICAL MONTHLY will contain a short statement concerning the scientific and editorial work of Professor F. N. Cole, who was, until the first of January of this year, the editor of this BULLETIN, and the secretary of the American Mathematical Society.

The October number of the BULLETIN OF THE NATIONAL RESEARCH COUNCIL (vol. 1, no. 5) consisted of a report on the quantum theory by Professor E. P. Adams. A few copies of this number are available for free distribution.

The Paris Academy of Sciences announces the award of the following prizes in pure and applied mathematics, in addition to those listed in the December BULLETIN (p. 139): the Grand prize of the mathematical sciences, to Ernest Esclangon, director of the Observatory at Strasbourg, for his memoir *New researches on quasi-periodic functions;* the Montyon prize for mechanics, to Stéphane Drzewiecki, for his work on the general theory of the helix, with special reference to aerial and marine propellers. The Academy announces the following problem for its Bordin prize, to be awarded in 1923: *To find all the cases in which the search for the surfaces admitting a given linear element leads to a partial differential equation of the second order that is integrable by the method of Darboux.* The Damoiseau prize was not awarded in 1920, no satisfactory memoir on the announced subject, *the figures of equilibrium of a rotating fluid mass subject to newtonian attraction* (see this BULLETIN, vol. 24, p. 316), having been received; the same problem is proposed again as the subject for 1923.

The Ackermann-Teubner memorial prize for 1920 was awarded to Professor Gustav Mie for his memoirs on the theory of matter which appeared in volumes 37, 39, and 40 of the ANNALEN DER PHYSIK.

The $5,000 prize offered through the SCIENTIFIC AMERICAN for a popular essay on the Einstein theories was awarded to Mr. L. Bolton, of London. The essay appears in the number of the SCIENTIFIC AMERICAN for February 5.

The Edison medal, awarded annually for work in electrical engineering by the American Institute of Electrical Engineers, will be presented this year to Professor M. I. Pupin, of Columbia University.

Announcement has been made at Brown University of the completion of the Nathaniel French Davis fund in honor of Professor Davis, now emeritus, who was for forty-one years a teacher of mathematics in that university. The fund amounts to ten thousand dollars, and the income is to supplement the regular library appropriations in purchasing mathematical books and periodicals for the mathematical seminary of Brown University.

A series of four lectures on *Four great geometers* (Archimedes, Galileo, Newton, and Maxwell) was given in November, 1920, by Mr. W. D. Eggar at Gresham College, London.

Professor Paul Painlevé, of the University of Paris, has returned from an educational mission to China. He has succeeded in arranging for an institute of the higher Chinese studies to be subsidized by the Chinese government in Paris, and for a branch of the University of Paris to be created in one of the Chinese universities and supported in part by the French government. An interesting account of these arrangements is given in the number of SCIENCE for February 4.

Professor Emile Borel, of the University of Paris, has been appointed honorary director of the Ecole normale supérieure.

Dr. David Owen, of Birkbeck College, has been appointed head of the department of physics and mathematics at the Sir John Cass Technical Institute, Aldgate, England.

Dr. P. Schafheitlin has been admitted as privat-docent at the technical school at Berlin.

Associate professor S. L. Boothroyd, of the department of mathematics and astronomy of the University of Washington, is spending the year at the Lick Observatory. He has been appointed to succeed Professor O. M. Leland as professor of astronomy and geodesy at Cornell University in September 1921.

Professor H. C. Wilson, of the department of mathematics and astronomy at Carleton College, has been granted leave of absence for the current academic year, part of which he will spend in research at Mt. Wilson Observatory. His position will be filled by Professor E. A. Fath, of Beloit College.

Associate professor Florence P. Lewis, of Goucher College, Baltimore, has been promoted to a full professorship of mathematics.

Dr. F. E. Wood, of the University of Chicago, has been appointed assistant professor of mathematics at the Michigan Agricultural College.

Dr. W. L. Crum, of Yale University, has been granted leave of absence for the second half of the present academic year to fill a temporary vacancy at Williams College as assistant professor of mathematics.

Dr. Teresa Cohen, of Johns Hopkins University, has been appointed instructor in mathematics at Pennsylvania State College.

Dr. Mary G. Haseman has been appointed instructor in mathematics at the University of Illinois.

Professor R. H. Weber, of the University of Rostock, died in August, 1920, at the age of forty-six years.

Dr. J. E. Clark, professor of mathematics in the Sheffield Scientific School of Yale University from 1873 to 1901, died January 3, 1921, in his eighty-ninth year. Professor Clark has been a member of the American Mathematical Society since its foundation.

Professor William Rinck, of Calvin College, Grand Rapids, Michigan, was killed in an automobile accident November 11, 1920, at the age of forty-three years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BAUDET (P. J. H.). Het limietbegrip. Rede uitgesproken bij de aanvaarding van het ambt van hoogleeraar aan de Technische Hoogeschool te Delft. Groningen, Noordhoff, 1919. 19 pp.

BROCARD (H.). See MALO (E.).

BUTLER (N. M.). See KLAPPER (P.).

CAREY (F. S.). Infinitesimal calculus. Re-issue. New York, Longmans, 1919. 10 + 352 pp. $4.25

GÉRARDIN (A.). See MALO (E.).

GRÜNBAUM (H.) und JACOBI (J.). Funktionenlehre und Elemente der Differential- und Integralrechnung. 4te Auflage. Leipzig, Teubner, 1920. 194 pp. M. 6.00

HARDY (G. H.). Some famous problems of the theory of numbers and in particular Waring's problem. An inaugural lecture delivered before the University of Oxford. Oxford, Clarendon Press, 1920. 36 pp. 8vo. Paper. 2s. 6d.

HEATH (T. L.). Pioneers of progress: Archimedes. ("Men of Science" Series.) London, Society for the Promotion of Christian Knowledge, 1920. 8vo. 2 + 58 pp. Cloth. 2s.

HUYGENS (C.). Œuvres complètes. Tome 14: Calcul des probabilités. Travaux de mathématiques purcs 1655–1666. The Hague, Nijhoff, 1920. 5 + 557 pp. 4to.

JACOBI (J.). See GRÜNBAUM (H.).

KLAPPER (P.). College teaching. Studies in methods of teaching in the college. Edited by P. Klapper with an introduction by N. M. Butler. [Chapter on the teaching of mathematics by G. A. Miller.] Yonkers, N. Y., World Book Company, 1920. 16 + 583 pp.

KNOPP (K.). Funktionentheorie. 2ter Teil. 2te Auflage. Berlin, Vereinigung wissenschaftlicher Verleger (Walter de Gruyter), 1920. 138 pp.

MALO (E.). Sur le décomposition des nombres en leurs facteurs. Résolution des équations numériques. Préfaces de A. Gérardin et de H. Brocard. Nancy, Sphinx-Œdipe, 1920. 128 + 48 pp.

MILLER (G. A.). See KLAPPER (P.).

OS (C. VAN). Iets over de ontwikkeling der meetkunde. Rede, uitgesproken bij de aanvaarding van het ambt van hoogleeraar aan de Technische Hoogeschool te Delft. Groningen, Noordhoff, 1919.

POLETTI (L.). Tavoli di numeri primi entro limiti diversi, e tavole affini. (Manuali Hoepli.) Milano, Hoepli, 1920. 16mo. 20 + 294 pp. L. 10.50

STUYVAERT (M.). Algèbre à deux dimensions. Gand, Van Rysselberghe et Rombaut, 1920. 8vo. 223 pp.

ZEUTHEN (G. H.). Sur l'origine de l'algèbre. København, Høst, 1919.

ZIEHEN (T.). Lehrbuch der Logik auf positivistischer Grundlage mit Berücksichtigung der Geschichte der Logik. Bonn, A. Marucs und E. Webers Verlag, 1920.

II. ELEMENTARY MATHEMATICS.

AMMERMAN (C.). See FORD (W. B.).

BALL (W. W. R.). An introduction to string figures. An amusement for everybody. Cambridge, W. Heffer and Sons, 1920. 8vo. 38 pp. Paper. 2s.

FORD (W. B.) and AMMERMAN (C.). Teachers' manual, second course in algebra. New York, Macmillan, 1920. 5 + 247 pp. $1.20

GIRNDT (M.). Raumlehre für Baugewerkschulen und verwandte bautechnische Schulen. 2ter Teil: Dreiecksberechnung und Körperlehre. 5te, neubearbeitete Auflage. Leipzig, Teubner, 1919.

HOLTON (E. E.). See PADDOCK (C. E.).

PADDOCK (C. E.) and HOLTON (E. E.). Vocational arithmetic. New York, Appleton, 1920. 10 + 232 pp.

RUDERT (G.). Die Grundlagen des funktionalen Denkens in ihrer Bedeutung für den ersten mathematischen Unterricht. Berlin, Salle, 1919. 106 pp. M. 3.50

THIJN (A. VAN). Leerboek der stereometrie. Vijfde druk. Groningen, J. B. Wolters, 1919.

III. APPLIED MATHEMATICS.

ALLIAUME (M.). Cours d'astronomie. Partie générale élémentaire. Louvain, Uystpruyst-Dieudonné, et Paris, Gauthier-Villars, 1920. 8vo. 14 + 368 pp.

BADER (H. G.). Grundlagen der Flugtechnik. Entwerfen und Berechnen von Flugzeugen. Leipzig, Teubner, 1920. 8 + 194 pp.

BAYLE (G.). Cours de résistance des matériaux appliquée aux machines. Paris, Librairie de l'Enseignement technique, 1920. 8vo. 478 pp. Fr. 36.00

BOUASSE (H.). Théorie de l'élasticité. Résistance des matériaux. Paris, Delagrave, 1920. 8vo. 562 pp.

BURGATTI (P.). Lezioni di meccanica razionale. 2a edizione. Bologna, Zanichelli, 1919. 8vo. 12 + 545 pp.

CARMICHAEL (R. D.). The theory of relativity. 2d edition. (Mathematical Monograph Series, No. 12.) New York, Wiley, 1920. 112 pp. $1.50

CREHORE (A. C.). The new physics. San Francisco, published by the *Journal of Electricity*, 1920. 128 pp. $2.00

DELAFONTAINE (F. L.). Abaques pour le calcul des dalles, poutres et poteaux en ciment armé. Paris et Liège, Béranger, 1920. 4to. 24 pp. + 18 abaques. Fr. 24.00

DISPER (P.). Theorie der Entwicklung der Kometen aus den Prinzipien der Gravitation. Montabour, W. Kalb, 1919.

EINSTEIN (A.). Relativity: the special and the general theory. A popular exposition. Authorized translation by R. W. Lawson. London, Methuen, and New York, Holt, 1920. 13 + 138 pp. 5s.

FONTVIOLANT (B. DE). Les méthodes modernes de la résistance des matériaux. 2e édition. Paris, Gauthier-Villars, 1920. 8vo. 104 pp. Fr. 9.00

FÜRTH (R.). Schwankungserscheinungen in der Physik. (Sammlung Vieweg, Nr. 48.) Braunschweig, Vieweg, 1920.

GIANFRANCESCHI (G.). La fisica dei corpuscoli. 2a edizione. Torino, Fratelli Bocca, 1920. 8vo. 264 pp.

GRÜBLER (M.). Getriebelehre. Eine Theorie des Zwanglaufes und der ebenen Mechanismen. Berlin, Springer, 1918.

HAREN (R.). Aufgabensammlung zur Festigkeitslehre mit Lösungen. 2te Auflage. Berlin, Vereinigung wissenschaftlicher Verleger (Walter de Gruyter), 1919. 89 pp.

KRAITCHIK (M.). Les tables graphiques financières. Paris, Gauthier-Villars, 1920. 4to. 20 pp. Fr. 7.50

LAWSON (R. W.). See EINSTEIN (A.).

MÜLLER (J.). Nautik. 2te Auflage. (Aus Natur und Geisteswelt, Nr. 255.) Leipzig, Teubner, 1919. 116 pp. Geb.

D'OCAGNE (M.). Principes usuels de nomographie avec application à divers problèmes concernant l'artillerie et l'aviation. Paris, Gauthier-Villars, 1920. 8vo. 4 + 70 pp. Fr. 9.00

RATEAU (A.). Théorie des hélices propulsives marines et aériennes et des avions en vol rectiligne. Paris, Gauthier-Villars, 1920. 8vo. 6 + 160 pp. Fr. 20.00

SAINTIGNON (F. DE). Le mouvement ininterrompu des molecules dans les fluides et autres hypothèses. Luxembourg, Imprimerie de la Cour Victor Buck, 1917. 4to. 175 pp.

SCHAU (A.). Statik. 1ter und 2ter Teil. 2te Auflage. Leipzig, Teubner, 1919.

VOLQUARDTS (G.). Feldmessen und Nivellieren. 4te Auflage. Leipzig, Teubner, 1919.